Journal of Statistical Physics, Vol. 19, No. 6, 1978

Exact Results for the Potts Model in Two Dimensions

A. Hintermann,[1] H. Kunz,[2,3] and F. Y. Wu[4,5]

Received August 18, 1978

By considering the zeros of the partition function, we establish the following results for the Potts model on the square, triangular, and honeycomb lattices: (i) We show that there exists only one phase transition; (ii) we give an exact determination of the critical point; (iii) we prove the exponential decay of the correlation functions, in one direction at least, for all temperatures above the critical point. The results are established for $q \geqslant 4$, where q is the number of components.

KEY WORDS: Potts model; critical point; zeros of partition function; correlation function.

1. INTRODUCTION

The critical point of the Potts model was first conjectured for the square lattice by Potts[1] using the Kramers–Wannier argument[2] in conjunction with an assumption of the uniqueness of the transition. The conjecture has since been extended to the triangular and honeycomb lattices[3,4] under more stringent conditions. While it has been established that a transition indeed occurs at these conjectured points,[4,5] the uniqueness of the transition has not been proven and, consequently, the determination of the Potts critical point remains very much an unsettled question.

We present in this paper an analysis of the Potts model which leads to an exact determination of its critical point for the square, triangular, and honeycomb lattices. In addition to confirming the previous conjectures, our analysis also establishes the uniqueness of the transition. We also prove that the correlation functions decay exponentially above the critical point.

[1] Swiss Institute for Nuclear Research (SIN), Villigen, Switzerland.
[2] Laboratoire de Physique Théorique, Ecole Polytechnique Fédérale, Lausanne, Switzerland.
[3] Work supported by the Fond. National Suisse de la Recherche Scientifique.
[4] Department of Physics, Northeastern University, Boston, Massachusetts.
[5] Work supported in part by NSF Grant No. DMR 76-20643.

0022-4715/78/1200-0623$05.00/0

Our analysis is based on the consideration of the zeros of the Potts partition function. Generally, the zeros of the partition function of a thermodynamic system trace a certain locus in the complex inverse temperature $\beta = 1/kT$ plane, and a phase transition occurs at the point where the locus crosses the positive β axis.[6] The strategy of our consideration is then to determine the region in the neighborhood of the positive β axis that is free of zeros. To carry out such an analysis we first convert the Potts model into a vertex model[5,7] for which some information on the zero distribution is known.[8] Since there is only one complex variable, namely β, arising in the Potts model, the fugacity z in the vertex model is actually temperature dependent. We next let z become independent and consider more generally the partition function $Z(\beta, z)$ of the vertex model. This permits the use of the circle theorem due to Suzuki and Fisher,[8] which states that $Z(\beta, z)$ is free of zeros for β real and $|z| \neq 1$. We further establish in the appendix that $Z(\beta, z)$ is free of zeros for small $|z|$ and β in a neighborhood of the positive axis. Combining these two results and making use of the Lebowitz–Penrose Lemma,[9] we then deduce that $Z(\beta, z)$ is in fact free of zeros for all $|z| \neq 1$ and β in a neighborhood of the positive axis. Returning now to the Potts model for which $z = z(\beta)$, this implies that the Potts free energy can be nonanalytic in β only at $|z(\beta)| = 1$; this in turn leads to a unique critical point. In Section 3 we sketch a proof which establishes the exponential decay of the correlation functions for all temperatures above the critical point.

Due to technical reasons, our results are established strictly only for $q \geqslant 4$, where q is the number of components in the Potts model. Since the $q = 2$ (Ising) model is exactly soluble, this leaves only the $q = 3$ Potts model unsettled for the time being.

2. POTTS CRITICAL POINT

It was established by Baxter *et al.*[7] that the partition function Z_N of the Potts model on a planar lattice L of N sites is related to the partition function Z_N' of an ice-rule vertex model on a related medial lattice L' by the relation

$$Z_N = q^{N/2} Z_N' \tag{1}$$

The medial lattice is essentially the covering lattice of L, constructed by connecting the midpoints of adjacent edges of L. For example, as illustrated in Figs. 7 and 8 of Ref. 7, the medial lattice of a square lattice is also square, and that of the triangular and honeycomb lattices is the Kagomé lattice. The vertex configurations of the ice-rule model are shown in Fig. 1a, where the

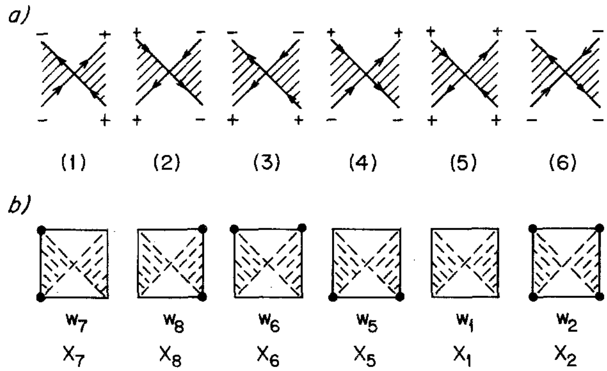

Fig. 1. (a) The ice-rule vertex configurations of the medial lattice and the associated spin configurations. The shaded regions denote the faces occupied by the sites of the Potts lattice. (b) The corresponding Ising configurations X_j and their weights w_j as defined in Ref. 17.

shaded regions denote the faces of L' occupied by the sites of L. In the notations of Ref. 7, the vertices have the following weights:

$$\{\omega_1, \ldots, \omega_6\} = \{1, 1, x_r, x_r, A_r, B_r\} \tag{2}$$

where

$$\begin{aligned}(A_r, B_r) &= (s^{-1} + x_r s, s + x_r s^{-1}) \quad \text{square} \\ &= (t^{-1} + x_r t^2, t + x_r t^{-2}) \quad \text{triangular}\end{aligned} \tag{3}$$

with

$$\begin{gathered} s = e^{\theta/2}, \qquad t = e^{\theta/3}, \qquad \cosh\theta = \sqrt{q}/2 \\ x_r = u_r/\sqrt{q}, \qquad u_r = \exp(\beta\epsilon_r) - 1 \end{gathered} \tag{4}$$

Here, $\epsilon_r > 0$ is the interaction in the Potts model between neighboring sites on edges of L in a given direction r, $r = 1, 2$ $(=1, 2, 3)$ for the square (triangular) lattice. There is no need to consider the honeycomb lattice separately since the triangular and honeycomb Potts models are related by a duality relation.(10) We have also included in Fig. 1a a spin representation of the vertex configurations, obtained by assigning a spin σ to each arrow such that $\sigma = +1$ (-1) if, crossing the arrow from the shaded region to the unshaded region, the arrow points toward one's right (left). We also remark that the vertex model (2) offers a natural extension of the Potts model to nonintegral values of q, which we shall assume to be the case.

Next we generalize the vertex weights (2) into a form reflecting the ice-rule restriction. Observe that if all arrows on the edges in a given direction are reversed, vertices (5) and (6) are converted to either the source or the sink of arrows. The conservation of arrows then implies the following relation:

$$n_{r5} + n_{r'6} = n_{r6} + n_{r'5}, \qquad r \neq r' \tag{5}$$

where n_{r5} (n_{r6}) denotes the number of the (5) [(6)] vertices on the type r edge of L. As a consequence of (5), the partition function Z_N' is unchanged if we use the following vertex weights in place of (2):

$$\begin{aligned}\{\omega_1,..., \omega_6\} &= \{1, 1, x_r, x_r, u_r A_r, u_r^{-1} B_r\} \\ &\equiv \{1, 1, x_r, x_r, c_r z, c_r z^{-1}\}\end{aligned} \tag{6}$$

provided that we take

$$\begin{aligned} u_1 u_2 &= 1 \qquad \text{square} \\ u_1 u_2 u_3 &= 1 \qquad \text{triangular}\end{aligned} \tag{7}$$

A similar argument leading to (6) can be found in Ref. 4.

Now, the variables c_r and z given by

$$c_r^2 = A_r B_r \tag{8}$$

$$\begin{aligned} z^4 &= A_1 A_2 / B_1 B_2 \qquad \text{square} \\ z^6 &= A_1 A_2 A_3 / B_1 B_2 B_3 \qquad \text{triangular}\end{aligned} \tag{9}$$

are both functions of the inverse temperature β of the Potts model. In order to make use of an established theorem on the zeros of a partition function, we now generalize the partition function Z_N' by regarding z in (6) as an independent variable and consider $Z_N' = Z_N'(\beta, z)$. Any conclusion so reached for $Z_N'(\beta, z)$ can obviously be specialized to the Potts model by once again introducing (9). Note that $Z_N'(\beta, z)$ is invariant under the change $z \to z^{-1}$, since a reversal of all arrows results in only an interchange of the weights ω_5 and ω_6 in (6).

To locate the zeros of $Z_N'(\beta, z)$ in the complex z plane for real β, we make use of a generalized Lee–Yang circle theorem due to Suzuki and Fisher.[8] Identifying z in (6) as the same variable z appearing in Eq. (2.3) of Ref. 8, and using the spin representation of the vertex configurations shown in Fig. 1a, we see that the partition function $Z_N'(\beta, z)$ is in precisely the form of that occurring in the Suzuki–Fisher (SF) theorem. For real β and

$$q > 4 \tag{10}$$

the variables x_r and c_r are both real, so that the condition (A) of the SF theorem is fulfilled. It is also readily verified that the condition (B) of the SF theorem holds under the same conditions. It then follows from the SF theorem that the zeros of $Z_N'(\beta, z)$ lie on the unit circle in the complex z plane for real β and $q > 4$.

Since $Z_N'(\beta, z) \sim |z|^{-M}$ for small $|z|$, where $M = 2N$ and $3N$, respectively, for the square and triangular lattices L, it proves convenient to consider, instead of Z_N', the function

$$F_N(\beta, z) = z^M Z_N'(\beta, z) \tag{11}$$

which is a polynomial of degree $2M$ in z. Using (6), we see in particular that

$$F_N(\beta, 0) = C^N \tag{12}$$

where

$$C = \prod_r c_r \tag{13}$$

This permits us to write

$$F_N(\beta, z) = C^N \prod_{i=1}^{2M} (1 - z/z_i) \tag{14}$$

where z_i are the $2M$ zeros of $F_N(\beta, z)$ satisfying $|z_i| = 1$ for real β and $q > 4$.

Consider now the function

$$G_N(\beta, z) = [F_N(\beta, z)]^{-1} \tag{15}$$

We have established that:

(i) $F_N(\beta, z) \neq 0$ for $|z| \neq 1$, β real, and $q > 4$.

We shall also establish in the appendix that:

(ii) $F_N(\beta, z) \neq 0$ for all $|z| < \delta$ and $\operatorname{Re}\beta \geqslant 0$, $|\operatorname{Im}\beta| < \pi/2\epsilon$, $q > 4$, where $\epsilon = \sup_r \epsilon_r$ and δ is some strictly positive constant depending only on $\{\epsilon_r\}$.

Furthermore, (14) implies the following bound on $G_N(\beta, z)$:

(iii)
$$\begin{aligned} G_N(\beta, z) &\leqslant C^{-N} \prod_i (1 - |z/z_i|) \\ &= C^{-N}(1 - |z|)^{-2M} \end{aligned} \tag{16}$$

The function $G_N(\beta, z)$ now satisfies precisely the conditions of the Lebowitz–Penrose Lemma[9] for a function of two variables. Applying the Lemma, we conclude that:

(iv) $F_N(\beta, z) \neq 0$ for all $|z| \neq 1$, $q > 4$, and β in some neighborhood of the positive real axis, the region of the neighborhood being uniform with regard to N.

Now $z(\beta)$ is real analytic in β. It follows from (iv) that for $q > 4$, the partition function $Z_N(\beta)$ of the Potts model is free of zeros in β, when β is in a complex neighborhood D of $[0, \beta_c)$ of $(\beta_c, \infty]$, where $\beta_c = \beta_c(q)$ is given by $z(\beta_c) = 1$. In fact, this conclusion holds also for $q = 4$, provided that we take

$$\beta_c(4) = \lim_{q \to 4+} \beta_c(q) \tag{17}$$

This is permitted because $Z_N(\beta)$ is a polynomial in e^β whose coefficients are continuous functions of q. Consequently, the zeros of the polynomial also depend on q continuously.

Following the standard arguments,[6] we now conclude that, for $q \geqslant 4$, the free energy of the Potts model,

$$f(\beta) = \lim_{N \to \infty} (1/M) \ln Z_N(\beta) \tag{18}$$

is a real analytic function of β when β is positive, except possibly at β_c. To determine β_c, we use (9) and the condition $z(\beta_c) = 1$ to obtain

$$\begin{aligned} x_1 x_2 &= 1 && \text{square} \\ \sqrt{q}\, x_1 x_2 x_3 + x_1 x_2 + x_2 x_3 + x_3 x_1 &= 1 && \text{triangular} \\ \sqrt{q} + x_1 + x_2 + x_3 &= x_1 x_2 x_3 && \text{honeycomb} \end{aligned} \tag{19}$$

Here we have used the duality relation[10] $x_\tau x_\tau^* = 1$ to relate the triangular and the honeycomb lattices.

Two comments are in order at this point. First we comment on the limitation of our results to $q \geqslant 4$. For $0 < q < 4$, conditions (A) and (B) of the Suzuki–Fisher theorem no longer hold and the locus of the zeros of $Z_N'(\beta, z)$ is not known. However, numerical results[11] indicate that the zeros do leave the unit circle, and, in fact, $z(\beta)$ lies on the unit circle for β real. It is clear that the strategy of the proof would be very different. This seems to confirm the change of the analytic properties of the Potts model found to exist at $q = 4$.[4,5] We wish to point out, however, that the critical point (19) does coincide with the exact (Ising) result at $q = 2$, and agrees with the previously conjectured critical point[3,4] including the $q = 1$ limit of the bond percolation.[12]

Finally, we comment that, strictly speaking, our analysis establishes only the fact that the nonanalyticity of $f(\beta)$, if any, can occur only at β_c. Now it has been explicitly established that $f(\beta)$ is indeed nonanalytic in β at β_c.[4,5] It follows that the Potts model has only one critical point, and that the critical point is given by (19).

3. CORRELATION FUNCTIONS

An interesting consequence of our analysis is that it allows us to establish the exponential decay of the correlation functions, for all temperatures above the critical temperature. We outline here the main steps of the proof.

First of all, our result on the zeros of the partition function remains true if, instead of using the free boundary conditions, we take a boundary condition such that the lattice is periodic in one direction. For the lattice is then still planar and all steps of our proof including the adoption of the result of Ref. 7 remain unchanged. In particular, a transfer matrix formulation of the partition function can be formulated, and there exists a domain D in the $u = e^{\beta} - 1$ plane containing the origin and the segment $[0, \exp(\beta_c) - 1]$ such that $Z_N(\beta) \neq 0$ when $u \in D$. Also, it has been shown by Israel[13] that when $|u| < c$, where c is some constant, $Z_N(\beta) \neq 0$ and the correlation functions decay exponentially. These two facts now permit the use of a theorem due to Penrose and Lebowitz[14] to conclude that the gap between the largest and the second largest eigenvalues of the transfer matrix remains nonzero, uniformly in N, when $u \in D$. But since this gap is a lower bound to the coherence length in the direction of periodicity along which the transfer matrix is defined, it follows that the correlation functions decay exponentially in the direction of periodicity for all $u \in D$, and hence for all $0 < \beta < \beta_c$. This establishes the stated result. Details of the proof follow closely that of Ref. 14 for the lattice gas, and will not be reproduced.

APPENDIX. PROOF OF PROPERTY (ii)

We establish in this appendix the property (ii) on the zeros of the function $F_N(\beta, z)$. The strategy here is to use the spin representation of the six-vertex model shown in Fig. 1a, and consider this as a constrained Ising model. The Asano contraction technique[15] is then applied to yield the desired property.

The idea of the Asano contraction is to obtain $F_N(\beta, z)$ by "contraction" of polynomials in few variables so as to relate the properties of zeros of the small polynomials to the zeros of $F_N(\beta, z)$. In the present case of a six-vertex model, the main problem of finding the small polynomials to build up $F_N(\beta, z)$ has already been solved in a more general context by Hintermann and Gruber.[16,17] The following discussion uses results established in Ref. 17.

The first step is to conform with the notations of Ref. 17. Associate a dot to each spin $\sigma = -1$ as indicated in Fig. 1b and compare the resulting configurations with those shown on p. 189 of Ref. 17. We then find the following relationships between the vertex weights w_j of Ref. 17 and the vertex weights ω_j defined in Section 2:

$$\begin{aligned}\{w_1, w_2, \ldots, w_8\} &= \{\omega_5, \omega_6, 0, 0, \omega_4, \omega_3, \omega_1, \omega_2\}, \qquad r = 1\\ &= \{\omega_5, \omega_6, 0, 0, \omega_1, \omega_2, \omega_3, \omega_4\}, \qquad r = 2 \qquad \text{(A1)}\\ &= \{\omega_5, \omega_6, 0, 0, \omega_3, \omega_4, \omega_2, \omega_1\}, \qquad r = 3\end{aligned}$$

where $r = 1, 2$ $(1, 2, 3)$ for the square (triangular) lattice.

Following Ref. 17, we put $w_j = \exp(-\beta e_j)$ and adopt the convention $e_j = 0$ if $w_j = 0, j = 1, 2, ..., 8$. Then from (6) we have

$$
\begin{aligned}
-\beta\{e_1, ..., e_8\} &= \{\ln c_1 z, \ln c_1 z^{-1}, 0, 0, \ln x_1, \ln x_1, 0, 0\}, \qquad r = 1 \\
&= \{\ln c_2 z, \ln c_2 z^{-1}, 0, 0, 0, 0, \ln x_2, \ln x_2\}, \qquad r = 2 \\
&= \{\ln c_3 z, \ln c_3 z^{-1}, 0, 0, \ln x_3, \ln x_3, 0, 0\}, \qquad r = 3
\end{aligned}
\tag{A2}
$$

Notice that the energies e_j for $r = 1$ and 3 are identical except for the difference in the subscripts. Next, as in the conventions given in p. 190 of Ref. 17, we associate to each vertex a local Hamiltonian $-H_c$ such that

$$H_c(X_j) = e_j, \qquad j = 1, 2, ..., 8 \tag{A3}$$

where X_j refers to the spin configurations. Write

$$-H_c = J_0 + \sum_{i=1}^{4} J_i \sigma_i + \tfrac{1}{2} \sum_{i \neq k = 1}^{4} J_{ik} \sigma_{ik} \tag{A4}$$

where $\sigma_1, ..., \sigma_4$ are the four spins surrounding a vertex and $\sigma_{ik} = \sigma_i \sigma_k$. We find with

$$
\begin{aligned}
J_B &= -\frac{1}{8} \sum_{j=1}^{8} \sigma_B(X_j) e_j \\
\sigma_B(X_j) &= (-1)^{|B \cap X_j|}, \qquad B \in \{(i), (i, k)\}_{i,k=1,...,4}
\end{aligned}
\tag{A5}
$$

$$
\begin{aligned}
&(4\beta)\{J_0, J_1 = J_2 = J_3 = J_4, J_{12} = J_{34}, J_{13} = J_{24}, J_{14} = J_{23}\} \\
&\quad = \{\ln c_1 x_1, \ln z, \ln c_1 x_1, \ln(c_1/x_1), \ln(c_1/x_1)\}, \qquad r = 1 \\
&\quad = \{\ln c_2 x_2, \ln z, \ln(c_2/x_2), \ln(c_2/x_2), \ln c_2 x_2\}, \qquad r = 2 \\
&\quad = \{\ln c_3 x_3, \ln z, \ln c_3 x_3, \ln(c_3/x_3), \ln(c_3/x_3)\}, \qquad r = 3
\end{aligned}
\tag{A6}
$$

We now have the identity

$$Z_N' = \tilde{Z}_N \tag{A7}$$

where $\tilde{Z}_N$ is the partition function of the constrained Ising model described by the Hamiltonian (A4)–(A6) and in which only the configurations with $w_j \neq 0$ are allowed. As in Ref. 17, since each lattice site of the Ising model belongs to two constraints, each of which giving a field contribution $J_i = (4\beta)^{-1} \ln z$, the field activity variable is simply $z_i = \exp(-4\beta J_i) = z^{-1}$, $i = 1, 2, 3, 4$. Similarly, the two-body activities are

$$z_{ij} = \exp(-2\beta J_{ij}) \in \{(c_r x_r)^{-1/2}, (x_r/c_r)^{1/2}\}$$

and we have

$$\tilde{Z}_N = \tilde{Z}_N(z, z_{ij}) \tag{A8}$$

The next step is to study $\tilde{Z}_N$ for independent z, c_r, x_r. This can be achieved by means of the Asano contractions of small polynomials.[15] Since, in the notations of Ref. 17, Eq. (A4) implies $\mathscr{B} \equiv \mathscr{B}_\infty$ and we have trivially $\overline{\mathscr{B}} \supset \mathscr{B}_\infty$, we can use the prescription of p. 233 of Ref. 17 to find the following small polynomial:

$$M_c = 1 + z_{14}z_{13}z_{24}z_{23}(z_1z_2 + z_3z_4) + z_{12}z_{13}z_{24}z_{34}(z_1z_4 + z_2z_3) + z_1z_2z_3z_4 \tag{A9}$$

associated with the constraints. Since only the variables z_i undergo one contraction and the two-body activities undergo no contractions, we can consider the products of the two-body activities as complex parameters. It is then necessary to study only the following type of local polynomial:

$$M_c(z_1, z_2, z_3, z_4) = 1 + u_1(z_1z_2 + z_3z_4) + u_2(z_1z_4 + z_2z_3) + z_1z_2z_3z_4 \tag{A10}$$

with

$$\begin{aligned} \{u_1, u_2\} &= \{x_1c_1^{-1}, c_1^{-1}\}, \qquad r = 1 \\ &= \{c_2^{-1}, x_2c_2^{-1}\}, \qquad r = 2 \\ &= \{x_3c_3^{-1}, c_3^{-1}\}, \qquad r = 3 \end{aligned} \tag{A11}$$

Let $u_1, u_2 \in \mathbb{C}$. Since

$$\min_{|z_i|,|z_j|<\rho} \operatorname{Re}(u_kz_iz_j) = -|u_k|\rho^2$$

we have

$$\operatorname{Re} M_c(z_1, z_2, z_3, z_4) \geqslant 1 - 2(|u_1| + |u_2|)\rho^2 - \rho^4$$

for $|z_i| < \rho$, $i = 1, 2, 3, 4$. It follows that

$$M_c(z_1, z_2, z_3, z_4) \neq 0$$

for

$$|z_i|^2 < [1 + (|u_1| + |u_2|)^2]^{1/2} - |u_1| - |u_2| \tag{A12}$$

The case in which we are interested concerns two independent complex variables z and β. A straightforward calculation shows that there exists a $\delta > 0$ such that

$$\delta < [1 + (|u_1| + |u_2|)^2]^{1/2} - |u_1| - |u_2|$$

whenever $\operatorname{Re}\beta \geqslant 0$, $|\operatorname{Im}\beta| \leqslant \pi/2\epsilon$, where $\epsilon = \sup_r \epsilon_r$ and $q > 4$. This establishes property (ii) for Z' and, consequently, for $F_N(\beta, z)$.

ACKNOWLEDGMENT

One of us (FYW) wishes to thank Prof. Ph. Choquard for his kind hospitality at Laboratoire de Physique Théorique, Ecole Polytechnique Fédérale, where part of this research was done.

REFERENCES

1. R. B. Potts, *Proc. Camb. Phil. Soc.* **48**:106 (1952).
2. H. A. Kramers and G. H. Wannier, *Phys. Rev.* **60**:252 (1941).
3. D. Kim and R. I. Joseph, *J. Phys. C* **7**:L167 (1974).
4. R. J. Baxter, H. N. V. Temperley, and S. E. Ashley, *Proc. Roy. Soc. Lond. A* **358**:535 (1978).
5. R. J. Baxter, *J. Phys. C* **6**:L445 (1973).
6. C. N. Yang and T. D. Lee, *Phys. Rev.* **87**:404 (1952).
7. R. J. Baxter, S. B. Kelland, and F. Y. Wu, *J. Phys. A* **9**:397 (1976).
8. M. Suzuki and M. E. Fisher, *J. Math. Phys.* **12**:235 (1971).
9. J. L. Lebowitz and O. Penrose, *Comm. Math. Phys.* **11**:99 (1968).
10. F. Y. Wu, *J. Math. Phys.* **18**:611 (1977).
11. K. S. Chang, S. Y. Wang, and F. Y. Wu, *Phys. Rev. A* **4**:2324 (1971).
12. M. F. Sykes and J. W. Essam, *J. Math. Phys.* **5**:1117 (1964).
13. R. B. Israel, *Comm. Math. Phys.* **50**:245 (1976).
14. O. Penrose and J. L. Lebowitz, *Comm. Math. Phys.* **39**:165 (1974).
15. T. Asano, *Phys. Rev. A* **4**:1409 (1970).
16. A. Hintermann and C. Gruber, *Physica* **84A**:101 (1976).
17. C. Gruber, A. Hintermann, and D. Merlini, in *Lecture Notes in Physics, Vol. 60,* J. Ehlers and K. Hepp, eds. (Springer-Verlag, 1977).